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LETTER TO THE EDITOR

Discussion of "Behaviour of braced and unbraced semi-rigid frames", *Int. J. Solids Structures*, Vol. 24, pp. 893–913 (1988)

Having read the interesting article, in which the authors use the simplified form of the excellent beam–column-element by Oran (1973), I feel that some enhancements and actual corrections are needed in order to make the beam-element well-behaved. To my knowledge this element has been previously introduced only with nondimensionalized forces and displacements. The intention here is to supplement the authors' text which uses ordinary forces and their relationships to element's displacements and make it consistent with Oran's theory.

The authors mention that their approximate stability functions s_i are to be preferred over the conventional stability functions, c_i. The Oran's incremental tangent stiffness matrix is based on the c_i functions, which are given later. The bowing functions b_i are derivatives of the stability functions c_i, and their accuracy is very important. Both s_i and c_i functions are dependent only on the dimensionless axial force parameter ρ. Particularly in slender structures the ρ-parameter may have values that cause s_i and c_i stability functions to have drastically different values, especially during unconverged Newton–Raphson iterations. My opinion is that s_i-functions should be used only when $|\rho| < 2$, but it is better to use the exact c_i-functions exclusively. There is apparently a typing error in eqn (5a), where the second term should be $(2\pi^2\rho/15)$.

The exact stability functions are

$$c_1 = \frac{\phi(\sin\phi - \phi\cos\phi)}{2(1-\cos\phi) - \phi\sin\phi} \tag{1}$$

$$c_2 = \frac{\phi(\phi - \sin\phi)}{2(1-\cos\phi) - \phi\sin\phi} \tag{2}$$

when $\rho < 0$ (beam is compressed) and

$$c_1 = \frac{\phi(\phi\cosh\phi - \sinh\phi)}{2(1-\cosh\phi) + \phi\sinh\phi} \tag{3}$$

$$c_2 = \frac{\phi(\sinh\phi - \phi)}{2(1-\cosh\phi) + \phi\sinh\phi} \tag{4}$$

when $\rho > 0$ (positive axial force).

The argument of the trigonometric and hyperbolic functions is

$$\phi = \pi\sqrt{|\rho|}. \tag{5}$$

Series expressions are to be used when $|\rho| < 0.1$ and they are (Kassimali, 1983)

$$c_1 \approx 4+\frac{2}{15}\pi^2\rho-\frac{11}{6300}\pi^4\rho^2+\frac{1}{27000}\pi^6\rho^3 \tag{6}$$

$$c_2 \approx 2-\frac{1}{30}\pi^2\rho+\frac{13}{12600}\pi^4\rho^2-\frac{11}{378000}\pi^6\rho^3. \tag{7}$$

The sign of the first bowing function b_1 in Lui and Chen's eqn (8a) is wrong and the correct expression is

$$b_1 = \frac{-(c_1+c_2)(c_2-2)}{8\pi^2\rho}. \tag{8}$$

This will make it positive as it should be. The erroneous tangent stiffness matrix will affect only the convergence properties in Newton–Raphson iteration, but the errors made in calculating the unbalanced element forces will remain in the final computational results.

The series expression to be used when $|\rho| < 0.1$, is given by

$$b_1 \approx \frac{1}{40}-\frac{1}{2800}\pi^2\rho+\frac{1}{168000}\pi^4\rho^2-\frac{37}{388080000}\pi^6\rho^3. \tag{9}$$

Having experienced some difficulties after using the simplified incremental basic stiffness relationship given by Lui and Chen, I eventually ended up using the original form given by Oran. The incremental force–displacement matrix given by Oran has been verified numerically by me and I have found that it is consistent (exact) with beam–column force–displacement relationships. When using the simplified incremental force–displacement matrix, you often have to use many elements to model one natural element, in order to avoid convergence difficulties. This is true especially with slender beams.

The incremental local force–displacement matrix is given below, using the following expressions.

$$G_1 = 2[(b_1+b_2)\theta_A+(b_1-b_2)\theta_B] \tag{10}$$

$$G_2 = 2[(b_1-b_2)\theta_A+(b_1+b_2)\theta_B] \tag{11}$$

$$H = \frac{I}{AL^2}-\frac{1}{\pi^2}[b'_1(\theta_A+\theta_B)^2+b'_2(\theta_A-\theta_B)^2]. \tag{12}$$

b'_i-Functions are derivatives of the bowing functions with respect to ρ.

$$b'_1 = -\frac{(b_1-b_2)(c_1+c_2)+2c_2b_1}{4\rho} \tag{13}$$

$$b'_2 = -\frac{\pi^2(16b_1b_2-b_1+b_2)}{4(c_1+c_2)}. \tag{14}$$

When $|\rho| < 0.1$ the series expression for b'_1, to be used, is

$$b'_1 \approx -\frac{1}{2800}\pi^2+\frac{1}{84000}\pi^4\rho-\frac{37}{129360000}\pi^6\rho^2. \tag{15}$$

All the series expressions given here are based on Kassimali's (1983) article, in which he extends the large deformation formulation of Oran to include elastic–perfectly plastic hinges.

The incremental force–displacement matrix is

$$\dot{\mathbf{k}}_c = \frac{EI}{L}\mathbf{k}. \tag{16}$$

The components are

$$k_{11} = c_1 + \frac{G_1^2}{H} \tag{17}$$

$$k_{22} = c_1 + \frac{G_2^2}{H} \tag{18}$$

$$k_{12} = k_{21} = c_2 + \frac{G_1 G_2}{H} \tag{19}$$

$$k_{13} = k_{31} = \frac{G_1}{HL} \tag{20}$$

$$k_{23} = k_{32} = \frac{G_2}{HL} \tag{21}$$

$$k_{33} = \frac{1}{HL^2}. \tag{22}$$

Added to the simplified force–displacement matrix components are the second term in H caused by element's curvature, and G_i^2/H, G_1G_2/H-terms. Note that it is not very easy to check the $\dot{\mathbf{k}}_c$-matrix analytically, but one can always do it by using the numerical differentiation of the beam–column force–displacement relationships.

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